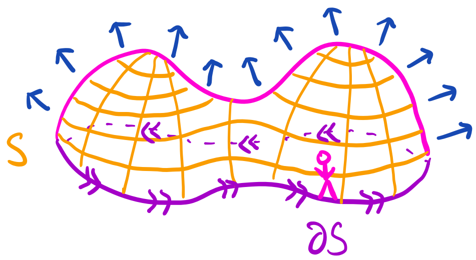
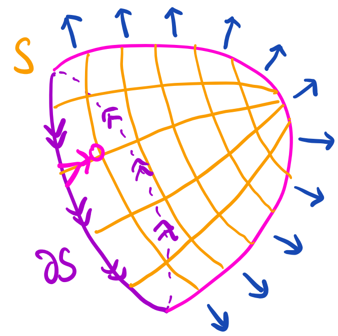
16.8. Stokes' theorem

Def Given a surface S, its boundary ∂S is positively oriented if it travels in a way that, when your head is aligned with the orientation of S, the interior of S lies on the left side.





Note If S does not contain any holes, then the positive orientation for ∂S is also given by the right hand rule.

Thm (Stokes' theorem)

Let $\vec{F}$ be a differentiable vector field on a surface S.

If the boundary ∂S is simple and positively oriented, then

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}.$$

Note (1) If $S = D$ is a domain in $\mathbb{R}^2$, then Stokes' theorem is equivalent to Green's theorem.

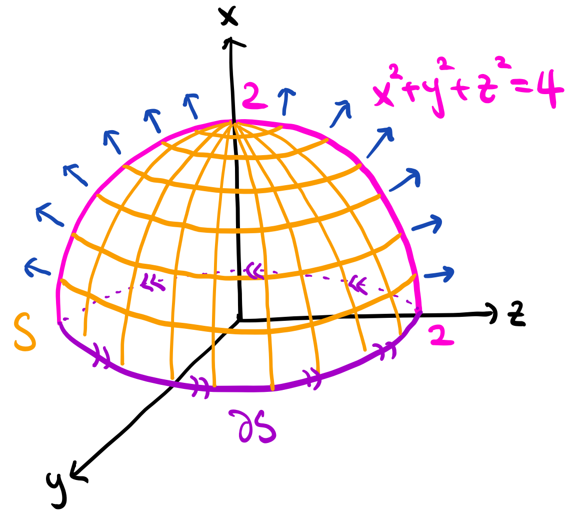
(2) Stokes' theorem is useful for

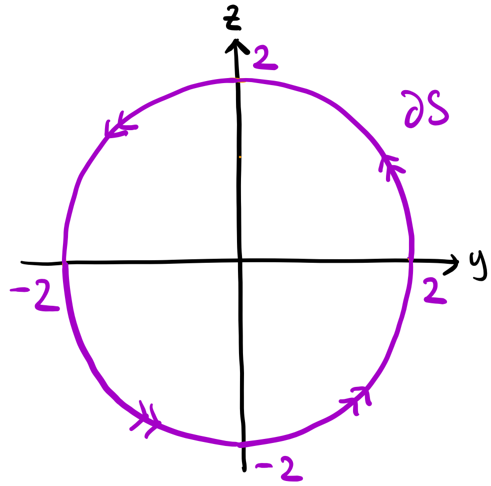
- computing $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$.
- computing $\int_C \vec{F} \cdot d\vec{r}$ where C is a loop in $\mathbb{R}^3$.

<u>Ex</u> Consider the vector field $\vec{F}(x,y,z) = (x^2+y^2,\ x-z,\ y)$.

(1) Find $\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$ where S is the hemisphere $x^2+y^2+z^2=4$ with $x \geq 0$, oriented in the positive direction of the x-axis.

<u>Sol</u>





$$\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} \underset{\uparrow}{=} \int_{\partial S} \vec{F}\cdot d\vec{r}$$

Stokes' thm

∂S is parametrized by $\vec{r}(t) = (0,\ 2\cos t,\ 2\sin t)$ on $0 \leq t \leq 2\pi$.

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$\vec{F}(\vec{r}(t)) = (4\cos^2 t,\ -2\sin t,\ 2\cos t)$$

$$\vec{r}\,'(t) = (0,\ -2\sin t,\ 2\cos t)$$

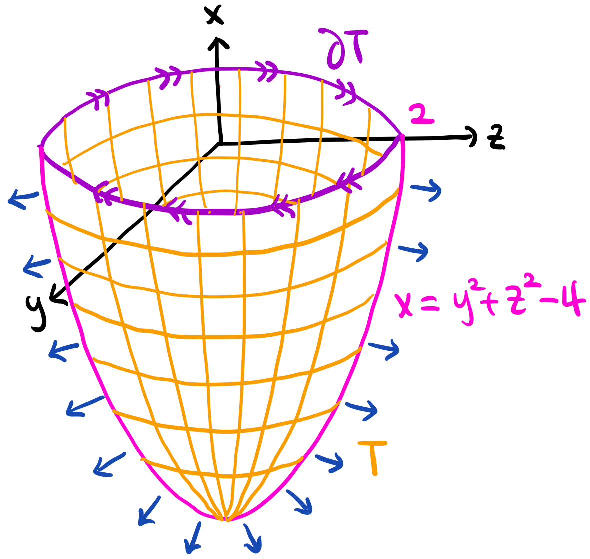
$$\Rightarrow \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 4\sin^2 t + 4\cos^2 t = 4.$$

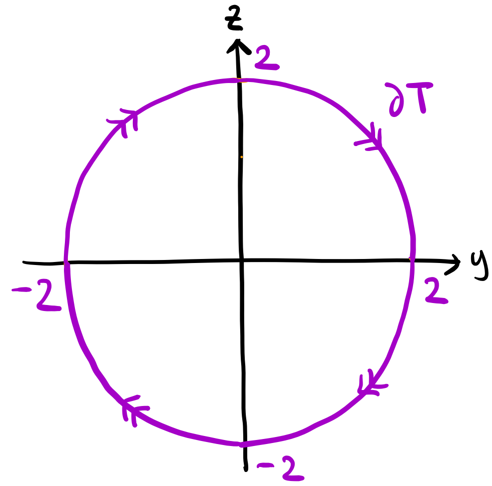
$$\Rightarrow \iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \int_0^{2\pi} 4\,dt = \boxed{8\pi}$$

<u>Note</u> This solution is very simple compared to a direct computation of the integral using a parametrization.

(2) Find $\iint_T \operatorname{curl}(\vec{F})\cdot d\vec{S}$ where T is the paraboloid $x = y^2+z^2-4$ with $x \le 0$, oriented in the negative direction of the x-axis.

Sol





$\partial T = -\partial S$ (opposite orientation)

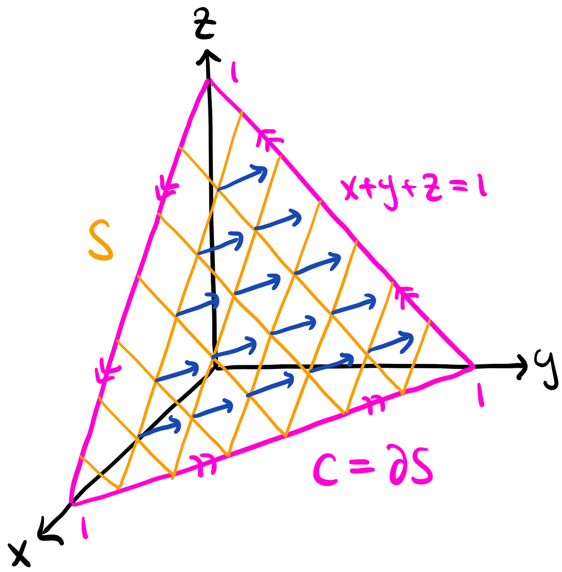
$$\iint_T \operatorname{curl}(\vec{F})\cdot d\vec{S} \underset{\text{Stokes' thm}}{=} \int_{\partial T} \vec{F}\cdot d\vec{r} = -\int_{\partial S} \vec{F}\cdot d\vec{r}$$

$$\underset{\text{Stokes' thm}}{=} -\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} \underset{(1)}{=} \boxed{-8\pi}$$

Note As you can see in this example, Stokes' theorem says that the surface integral of $\operatorname{curl}(\vec{F})$ does not depend on the shape of the surface as long as the boundary remains the same.

Ex Let C be the triangular loop that passes through the vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in order. Find the work done by the force field $\vec{F}(x,y,z) = (x+y^2, y+z^2, z+x^2)$ along the curve C.

Sol



S : the triangular surface bounded by C, oriented upward.

$\Rightarrow \partial S = C$ is positively oriented.

$\Rightarrow \int_{\partial S} \vec{F}\cdot d\vec{r} \underset{\text{Stokes' thm}}{=} \iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$.

S is a part of the plane $x+y+z=1 \rightsquigarrow z = 1-x-y$.

(The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ has x,y,z intercepts a,b,c)

$\Rightarrow S$ is parametrized by $\vec{r}(x,y) = (x, y, 1-x-y)$

The domain D is given by the shadow on the xy-plane

$\vec{r}_x = (1,0,-1)$, $\vec{r}_y = (0,1,-1)$

$\Rightarrow \vec{r}_x \times \vec{r}_y = (1,1,\underset{0}{\underset{\vee}{1}})$: oriented upward

$\vec{F}(x,y,z) = (x+y^2, y+z^2, z+x^2) \Rightarrow \operatorname{curl}(\vec{F}) = (-2z, -2x, -2y)$

$\operatorname{curl}(\vec{F})(\vec{r}(x,y))\cdot(\vec{r}_x \times \vec{r}_y) = (-2(1-x-y), -2x, -2y)\cdot(1,1,1) = -2$.

$$\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \iint_D \operatorname{curl}(\vec{F})(\vec{r}(x,y))\cdot(\vec{r}_x \times \vec{r}_y)\, dA$$

$$= \iint_D -2\, dA = -2\operatorname{Area}(D) = -2\cdot\tfrac{1}{2}\cdot 1\cdot 1 = \boxed{-1}$$